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Ectoplasm and superspace integration measures for 2D supergravity with four spinorial supercurrents

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Abstract

Building on a previous derivation of the local chiral projector for a two-dimensional superspace with eight real supercharges, we provide the complete density projection formula required for locally supersymmetrical theories in this context. The derivation of this result is shown to be very efficient using techniques based on the ectoplasmic construction of local measures in superspace.

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1. Introduction

Some years ago, a formulation of a 2D supergravity theory which included off-shell closure of the local supersymmetry algebra with four real spinorial supercharges and a necessary set of auxiliary fields was introduced into the literature [1]. In a subsequent development, a proposal was made (called 'ectoplasm' [2]) for a conceptual framework leading to efficient derivations of local superspace integration measures (density projection operators) [1]. In addition, at about the same time there was put forward an alternative general framework for the derivation of density projection operators based on the use of superspace normal coordinate expansions first introduced in [6] and rediscovered[2] in [7]; see [15] for recent reformulations and improvements of the normal coordinates techniques. The ectoplasm and normal coordinates frameworks have been found to be closely related [16, 17].

Prior to the introduction of the ectoplasmic and normal coordinate approaches, the question of how to construct local superspace supergravity densities had been approached by two other and more cumbersome methods. Both of these can be seen in two books on the subject. In

[1] A mathematical construction giving the formal bases for the ectoplasm methods can be found in the theory of integration over surfaces in supermanifolds developed in [3–5].

[2] Previous approaches for component reduction, ultimately related to normal coordinates expansions, can be found in [8–14].

the first, *Superspace* [18], an approach was taken to reproduce, at the level of superfields, a Noether approach thus leading to the density projector. In the second *Ideas* [19], an approach was taken to utilize the prepotential formulation of supergravity theory to derive the density projector.

It has been argued from its inception that the ectoplasmic concept is not only extremely efficient but also likely to apply to even more complicated theories such as string theory. Though there was no such evidence at the time of the introduction of the ectoplasm approach, later it was shown that integration measures in the 'pure spinor formulation' of superstrings follow precisely from the extension of the ectoplasmic concept to this realm of theories [20].

The off-shell formulation of a 2D, $\mathcal{N} = 4$ supergravity theory implies the existence of a straightforward way to completely develop an *efficient* local integration theory for the associated local Salam–Strathdee superspace. We will complete such a construction in the current work by use of the ectoplasmic suggestion.

This paper is organized as follows. In section 2 we review the 2D, $\mathcal{N} = 4$ supergravity formulation of [1]. Section 3 is devoted to the presentation of a new super two-form multiplet. In section 4 we make use of the ectoplasmic approach to build the density projector for the 2D, $\mathcal{N} = 4$ supergravity of [1]; this is the main result of the paper. Section 5 collects some conclusions. The paper includes two appendices. Appendix A contains the derivation of the result of section 3. Then, appendix B is a collection of formulas used in the paper.

2. An off-shell 2D supergravity geometry with eight real local supersymmetries

In this section we review some aspects of the off-shell 2D, $\mathcal{N} = 4$ minimal supergravity multiplet first introduced in [1]. We focus on the curved superspace geometry underlining the minimal supergravity that will be used in the computations of this paper.

The work in [1] showed there exists component fields $(e_a{}^m, \psi_a{}^{\alpha i}, A_{ai}{}^j, B, G, H)$ which describe an off-shell 2D supergravity theory possessing eight real local (or four real spinorial) supercharges. The previous list of component fields contains the graviton, the gravitini, $SU(2)$ connection, a complex scalar B, one real scalar G and one real pseudoscalar H. These are the components associated with the following constraints on the 2D, $\mathcal{N} = 4$ superspace supergravity covariant derivative algebra[3]:

$$\{\nabla_{\alpha i}, \nabla_{\beta j}\} = 2\overline{B}[C_{\alpha\beta}C_{ij}\mathcal{M} - (\gamma^3)_{\alpha\beta}\mathcal{Y}_{ij}], \tag{1}$$

$$\left\{\overline{\nabla}_\alpha{}^i, \overline{\nabla}_\beta{}^j\right\} = 2B[C_{\alpha\beta}C^{ij}\mathcal{M} - (\gamma^3)_{\alpha\beta}\mathcal{Y}^{ij}], \tag{2}$$

$$\left\{\nabla_{\alpha i}, \overline{\nabla}_\beta{}^j\right\} = 2\mathrm{i}\delta_i{}^j(\gamma^c)_{\alpha\beta}\nabla_c + 2\delta_i{}^j\phi_\alpha{}^\gamma(\gamma^3)_{\gamma\beta}\mathcal{M} - 2\phi_{\alpha\beta}\mathcal{Y}_i{}^j, \tag{3}$$

$$[\nabla_{\alpha i}, \nabla_b] = \frac{\mathrm{i}}{2}\phi_\alpha{}^\gamma(\gamma_b)_\gamma{}^\beta\nabla_{\beta i} + \frac{\mathrm{i}}{2}(\gamma^3\gamma_b)_\alpha{}^\beta\overline{B}C_{ij}\overline{\nabla}_\beta{}^j - \mathrm{i}(\gamma^3\gamma_b)_{\alpha\beta}\bar{\Sigma}^\beta{}_i\mathcal{M} + \mathrm{i}(\gamma_b)_{\alpha\beta}\bar{\Sigma}^\beta{}_j\mathcal{Y}_i{}^j, \tag{4}$$

$$\left[\overline{\nabla}_\alpha{}^i, \nabla_b\right] = -\frac{\mathrm{i}}{2}\bar{\phi}_\alpha{}^\gamma(\gamma_b)_\gamma{}^\beta\overline{\nabla}_\beta{}^i + \frac{\mathrm{i}}{2}(\gamma^3\gamma_b)_\alpha{}^\beta BC^{ij}\nabla_{\beta j} - \mathrm{i}(\gamma^3\gamma_b)_{\alpha\beta}\Sigma^{\beta i}\mathcal{M} - \mathrm{i}(\gamma_b)_{\alpha\beta}\Sigma^{\beta j}\mathcal{Y}^i{}_j, \tag{5}$$

$$[\nabla_a, \nabla_b] = -\tfrac{1}{2}\varepsilon_{ab}[(\gamma^3)_\alpha{}^\beta\Sigma^{\alpha i}\nabla_{\beta i} + (\gamma^3)_\alpha{}^\beta\overline{\Sigma}^\alpha{}_i\overline{\nabla}_\beta{}^i + \mathcal{R}\mathcal{M} + \mathrm{i}\mathcal{F}_i{}^j\mathcal{Y}_j{}^i], \tag{6}$$

[3] In the present paper we adopt the Lorentz and $SU(2)$ notations collected in appendix A of [21] and consistent with the conventions of [18].

where

$$(B)^* = \bar{B}, \qquad (G)^* = G, \qquad (H)^* = H, \qquad \left(\Sigma_\alpha{}^i\right)^* = \bar{\Sigma}_{\alpha i}, \tag{7}$$

$$\phi_{\alpha\beta} = C_{\alpha\beta} G + \mathrm{i}(\gamma^3)_{\alpha\beta} H, \tag{8}$$

$$\bar{\phi}_{\alpha\beta} = (\phi_{\alpha\beta})^* = -C_{\alpha\beta} G + \mathrm{i}(\gamma^3)_{\alpha\beta} H = \phi_{\beta\alpha}. \tag{9}$$

In writing these, we have corrected some coefficients that appear in the work of [21] in the terms that appear in (3)–(5). These corrected coefficients do not affect (1) and (2). Thus, the result in the work of [21] is unaffected by this change.

In the previous algebra, the covariant derivatives are $\nabla_A = \left(\nabla_a, \nabla_{\alpha i}, \overline{\nabla}_\alpha{}^i\right)$

$$\nabla_A = E_A{}^M \partial_M + \omega_A \mathcal{M} + \mathrm{i}\Gamma_{Ak}{}^l \mathcal{Y}_l{}^k. \tag{10}$$

The 2D, $\mathcal{N} = 4$ curved superspace is locally parametrized by the coordinates $z^M = (x^m, \theta^{\mu i}, \bar{\theta}^\mu{}_i)$ with the Grassmann variables $\theta^{\mu i}$ and $\bar{\theta}^\mu{}_i$ related by complex conjugation $\bar{\theta}^\mu{}_i = (\theta^{\mu i})^*$; the bosonic coordinates will be also denoted as $x^m = (\tau, \sigma)$. In (10), $E_A{}^M$ is the inverse of the vielbein $E_M{}^A$ $\left(E_M{}^A E_A{}^N = \delta_M^N, E_A{}^M E_M{}^B = \delta_A^B\right)$ with $\partial_M = \partial/\partial z^M$, ω_A the 2D Lorentz connection and $\Gamma_{Ak}{}^l$ is the $SU(2)$ connection. The torsion $T_{AB}{}^C$, Lorentz curvature R_{AB} and $SU(2)$ curvature $R_{ABk}{}^l$ superfields are defined by (1)–(6) and

$$[\nabla_A, \nabla_B\} = T_{AB}{}^C \nabla_C + R_{AB} \mathcal{M} + \mathrm{i} R_{ABk}{}^l \mathcal{Y}_l{}^k. \tag{11}$$

The action of the local 2D Lorentz generator $\mathcal{M}$ and of the local $SU(2)$ generator $\mathcal{Y}_k{}^l$ on the spinor covariant derivatives are the following ($\mathcal{Y}_{kl} = \mathcal{Y}_k{}^p C_{pl}$):

$$[\mathcal{M}, \nabla_{\alpha i}] = \tfrac{1}{2}(\gamma^3)_\alpha{}^\beta \nabla_{\beta i}, \qquad \left[\mathcal{M}, \overline{\nabla}_\alpha{}^i\right] = \tfrac{1}{2}(\gamma^3)_\alpha{}^\beta \overline{\nabla}_\beta{}^i, \tag{12}$$

$$[\mathcal{Y}_{kl}, \nabla_{\alpha i}] = \tfrac{1}{2} C_{i(k} \nabla_{\beta l)}, \qquad \left[\mathcal{Y}_{kl}, \overline{\nabla}_\alpha{}^i\right] = -\tfrac{1}{2}\delta^i_{(k} \overline{\nabla}_{\beta l)}. \tag{13}$$

It is worthy to recall that the consistency of the Bianchi identities constructed from the commutator algebra above requires the conditions [1]:

$$\overline{\nabla}_\alpha{}^i B = 0, \qquad \nabla_{\alpha i} B = -2C_{ij}(\gamma^3)_{\alpha\beta} \Sigma^{\beta j}, \tag{14}$$

$$\nabla_{\alpha i} G = \overline{\Sigma}_{\alpha i}, \qquad \nabla_{\alpha i} H = \mathrm{i}(\gamma^3)_\alpha{}^\beta \overline{\Sigma}_{\beta i}, \tag{15}$$

$$\overline{\nabla}_\alpha{}^i \Sigma^{\beta j} = \mathrm{i} C^{ij} (\gamma^3 \gamma^a)_\alpha{}^\beta \nabla_a B, \tag{16}$$

$$\nabla_{\alpha i} \Sigma^{\beta j} = \tfrac{1}{2}\delta_\alpha{}^\beta \delta_i{}^j [\mathcal{R} - 2G^2 - 2H^2 - 2B\overline{B}] + \mathrm{i}(\gamma^3)_\alpha{}^\beta \mathcal{F}_i{}^j \tag{17}$$

$$+ \mathrm{i}\delta_i{}^j (\gamma^a)_\alpha{}^\beta (\nabla_a G) - \delta_i{}^j (\gamma^3 \gamma^a)_\alpha{}^\beta (\nabla_a H). \tag{18}$$

The component gauge fields occur in the above supertensors in the following manner[4]:

$$\begin{aligned}
\mathcal{R}| = \varepsilon^{ab}\{&\mathcal{R}_{ab}(\hat{\omega}) + [2\mathrm{i}(\gamma^3 \gamma_a)_{\alpha\beta} \psi_b{}^{\alpha i} \overline{\Sigma}^\beta{}_i + \text{h.c.}] \\
&+ 4\phi_\alpha{}^\gamma (\gamma^3)_{\gamma\beta} \psi_a{}^{\alpha i} \overline{\psi}_b{}^\beta{}_i - 2[C_{ij} \overline{B} \psi_a{}^{\alpha i} \psi_{b\alpha}{}^j + \text{h.c.}]\}, \\
\Sigma^{\alpha i}| = \varepsilon^{ab}\big\{&\psi_{ab}{}^{\beta i}(\gamma^3)_\beta{}^\alpha - \mathrm{i}\psi_a{}^{\beta i} \phi_\beta{}^\gamma (\gamma^3 \gamma_b)_\gamma{}^\alpha + \mathrm{i} C^{ij} B \overline{\psi}_a{}^\beta{}_j (\gamma_b)_\beta{}^\alpha\big\}, \\
\mathcal{F}_i{}^j| = \varepsilon^{ab}\big\{&\mathrm{F}_{ab}(A)_i{}^j - 2\mathrm{i}(\gamma_a)_{\alpha\beta}\big[\psi_b{}^{\alpha j} \overline{\Sigma}^\beta{}_i + \overline{\psi}_b{}^\alpha{}_i \Sigma^{\beta j} - \tfrac{1}{2}\delta_i^j \big(\psi_b{}^{\alpha k} \overline{\Sigma}^\beta{}_k + \overline{\psi}_b{}^\alpha{}_k \Sigma^{\beta k}\big)\big] \\
&- 4\phi_{\alpha\beta}\big[\psi_a{}^{\alpha j} \overline{\psi}_b{}^\beta{}_i - \tfrac{1}{2}\delta_i^j \psi_a{}^{\alpha k} \overline{\psi}_b{}^\beta{}_k\big] \\
&- 2(\gamma^3)_{\alpha\beta}\big[\overline{B}\big(C_{ik} \psi_a{}^{\alpha k} \psi_b{}^{\beta k} - \tfrac{1}{2}\delta_i^j C_{kl} \psi_a{}^{\alpha k} \psi_b{}^{\beta l}\big) \\
&+ B\big(C^{jk} \overline{\psi}_a{}^\alpha{}_i \overline{\psi}_b{}^\beta{}_k - \tfrac{1}{2}\delta_i^j C^{kl} \overline{\psi}_a{}^\alpha{}_k \overline{\psi}_b{}^\beta{}_l\big)\big]\big\},
\end{aligned} \tag{19}$$

[4] Given a superfield $\Psi(\tau, \sigma, \theta, \bar{\theta})$, we denote as usual with $\Psi| := \Psi|_{\theta=0}$ the field obtained by setting to zero all the Grassmanian coordinates.

where $\varepsilon^{ab}\,\mathcal{R}_{ab}(\hat{\omega})$ is the usual two-dimensional curvature in terms of the inverse of the vielbein $e_a{}^m$ and of the Lorentz connection $\hat{\omega}_a$; $\varepsilon^{ab}\psi_{ab}{}^{\beta i}$ is the gravitini field strength; $\varepsilon^{ab}F_{ab}(A)$ is the $SU(2)$ field strength function of $e_a{}^m$ and of the $SU(2)$ connection $A_{ak}{}^l$ [1]. The component gauge fields $e_a{}^m$, $\hat{\omega}_a$, $A_{ak}{}^l$ are easily related to the gauge superfields $E_A{}^M$, ω_A, $\Gamma_{Ak}{}^l$ in (10) by using standard Wess–Zumino gauge reduction techniques [18, 19].

3. Defining a closed 2D, $\mathcal{N} = 4$ super two-form

In this section we are going to present a new closed 2D, $\mathcal{N} = 4$ super two-form defined in terms of an unconstrained scalar chiral superfield. The result contained in theorem 1 is crucial to build the measure of the local superspace integration theory for 2D, $\mathcal{N} = 4$ supergravity theories as we will see in section 4.

The work in [21] established that the fourth-order spinorial derivatives operator $\overline{\mathcal{D}}^{(4)}$, defined by

$$\overline{\mathcal{D}}^{(4)} = [\overline{\nabla}^{(2)\alpha\beta} - 2B(\gamma^3)^{\alpha\beta}]\overline{\nabla}^{(2)}_{\alpha\beta}, \tag{20}$$

is the chiral projection operator satisfying

$$\overline{\nabla}^i_\gamma\overline{\mathcal{D}}^{(4)}\Psi = \overline{\nabla}^i_\gamma[\overline{\nabla}^{(2)\alpha\beta} - 2B(\gamma^3)^{\alpha\beta}]\overline{\nabla}^{(2)}_{\alpha\beta}\Psi = 0 \tag{21}$$

for any general scalar superfield Ψ. We note that the derivation of (20) and (21) given in [21] follows solely from algebraic manipulations of the derivatives that appear in (2).

In a later section we will exploit the fact that a closed 2D, $\mathcal{N} = 4$ super two-form is sufficient to determine the local integration measure for an appropriate curved superspace. For this purpose it is necessary to define the components of a 2D, $\mathcal{N} = 4$ super two-form. The general framework for the construction of such forms was presented some time ago [22] which implies for the present consideration we should introduce a super two-form whose component superfields may be written in the form $J_{AB} = \big(J_{\alpha i\,\beta j}, J_{\alpha i\,\beta}{}^j, J_\alpha{}^i{}_\beta{}^j, J_{\gamma ka}, J_\gamma{}^k{}_a, J_{ab}\big)$. We refer the reader to [18, 22] for the notations we adopt in the use of super p-forms. In general, given a super p-form Ω, described by the component superfields $\Omega_{A_1\cdots A_p}$, its exterior derivative $F = \mathrm{d}\Omega$ has components $F_{A_1\cdots A_{p+1}}$ given by [5]

$$F_{A_1\cdots A_pA_{p+1}} = \frac{1}{p!}\nabla_{[A_1}\Omega_{A_2\cdots A_{p+1})} - \frac{1}{2((p-1)!)}T_{[A_1A_2|}{}^B\Omega_{B|A_3\cdots A_{p+1})}. \tag{22}$$

The superform Ω is closed if $F_{A_1\cdots A_{p+1}} = 0$. We can now state a theorem.

Theorem 1. *If U is a chiral superfield, i.e. satisfies $\overline{\nabla}^i_\alpha U = 0$, the components defined by*

$$\begin{aligned}
J_{\alpha i\,\beta}{}^j &= 0,\\
J_{\alpha i\,\beta j} &= 2(\gamma^3)_{\alpha\beta}\overline{\nabla}^{(2)}_{ij}\overline{U} - C_{\alpha\beta}C_{ij}(\gamma^3)^{\gamma\delta}\overline{\nabla}^{(2)}_{\gamma\delta}\overline{U},\\
J_\alpha{}^i{}_\beta{}^j &= 2(\gamma^3)_{\alpha\beta}\nabla^{(2)ij}U - C_{\alpha\beta}C^{ij}(\gamma^3)^{\gamma\delta}\nabla^{(2)}_{\gamma\delta}U,\\
J_{\gamma ka} &= -\frac{\mathrm{i}}{3}\varepsilon_{ab}(\gamma^b)_\gamma{}^\delta\overline{\nabla}_\delta{}^p\overline{\nabla}^{(2)}_{kp}\overline{U},\\
J_\gamma{}^k{}_a &= -\frac{\mathrm{i}}{3}\varepsilon_{ab}(\gamma^b)_\gamma{}^\delta\nabla_{\delta p}\nabla^{(2)kp}U,\\
J_{ab} &= -\frac{1}{8}\varepsilon_{ab}\big[\big(\nabla^{(4)} - 2\bar{B}(\gamma^3)^{\alpha\beta}\nabla^{(2)}_{\alpha\beta}\big)U + \big(\overline{\nabla}^{(4)} - 2B(\gamma^3)^{\alpha\beta}\overline{\nabla}^{(2)}_{\alpha\beta}\big)\overline{U}\big],
\end{aligned} \tag{23}$$

describe a closed 2D, $\mathcal{N} = 4$ super two-form with respect to the supergravity commutator algebra in equations (1)–(6).

[5] With $[\cdots)$ we denote the complete graded symmetrization of indices.

The superfield $\bar{U} := (U)^*$ is antichiral $\nabla_{\alpha i}\bar{U} = 0$. In writing these results, we have introduced second- and fourth-order spinorial derivative operators via the equations

$$\begin{aligned}
&\nabla^{(2)}_{\alpha\beta} = \tfrac{1}{2}C^{ij}[\nabla_{\alpha i}\nabla_{\beta j} + \nabla_{\beta i}\nabla_{\alpha j}], \qquad \nabla^{(2)}_{ij} = \tfrac{1}{2}C^{\alpha\beta}\left[\nabla_{\alpha i}\nabla_{\beta j} + \nabla_{\alpha j}\nabla_{\beta i}\right],\\
&\overline{\nabla}^{(2)}_{\alpha\beta} = \tfrac{1}{2}C_{ij}\left[\overline{\nabla}_{\alpha}{}^{i}\overline{\nabla}_{\beta}{}^{j} + \overline{\nabla}_{\beta}{}^{i}\overline{\nabla}_{\alpha}{}^{j}\right], \qquad \overline{\nabla}^{(2)ij} = \tfrac{1}{2}C^{\alpha\beta}\left[\overline{\nabla}_{\alpha}{}^{i}\overline{\nabla}_{\beta}{}^{j} + \overline{\nabla}_{\alpha}{}^{j}\overline{\nabla}_{\beta}{}^{i}\right],\\
&\nabla^{(4)} = \tfrac{1}{3}\nabla^{(2)kl}\nabla^{(2)}_{kl}, \qquad \overline{\nabla}^{(4)} = \tfrac{1}{3}\overline{\nabla}^{(2)kl}\overline{\nabla}^{(2)}_{kl}.
\end{aligned} \tag{24}$$

The proof of the theorem involves using the above equations to show that the Bianchi identities for this two-form vanish. This is relegated to an appendix. We next note that the chiral superfield U above may be replaced using the result from (21) according to $U = \bar{\mathcal{D}}^{(4)}\mathcal{L}$ ($\bar{U} = \mathcal{D}^{(4)}\bar{\mathcal{L}} = (\bar{\mathcal{D}}^{(4)}\mathcal{L})^*$) where the 2D, $\mathcal{N} = 4$ superfield $\mathcal{L}$, ($\bar{\mathcal{L}} := (\mathcal{L})^*$), is not subject to any algebraic nor differential restrictions. Stated another way, this implies that an arbitrary 2D, $\mathcal{N} = 4$ superfield $\mathcal{L}$ can be used to create a closed super two-form whose components are defined by J_{AB} above. We conclude with a result that will be needed in the next section. Defining the component vierbein $E_m{}^a| = e_m{}^a$ $\left(e_m{}^a e_a{}^n = \delta^n_m,\, e_a{}^m e_m{}^b = \delta^b_a\right)$, and the gravitini $E_m{}^{\alpha i}| = -\psi_m{}^{\alpha i}$ $\left(\psi_a{}^{\alpha i} = e_a{}^m\psi_m{}^{\alpha i}\right)$, $E_m{}^{\alpha}_i\big| = -\bar{\psi}_m{}^{\alpha}_i$ $\left(\bar{\psi}_a{}^{\alpha}_i = e_a{}^m\bar{\psi}_m{}^{\alpha}_i\right)$, by a general result given in [2, 18] taking the limit as all Grassmann coordinates go to zero one obtains

$$\begin{aligned}
\varepsilon^{ab}J_{ab}| = \varepsilon^{ab}\mathcal{J}_{ab}| &+ 2\varepsilon^{ab}\left(\psi_a{}^{\alpha i}J_{\alpha i b}\big| + \bar{\psi}_a{}^{\alpha}{}_i J_{\alpha}{}^{i}{}_b\big|\right) + 2\varepsilon^{ab}\psi_a{}^{\alpha i}\bar{\psi}_b{}^{\beta}{}_j J_{\alpha i\beta}{}^{j}\big|\\
&+ \varepsilon^{ab}\psi_a{}^{\alpha i}\psi_b{}^{\beta j}J_{\alpha i\beta j}\big| + \varepsilon^{ab}\bar{\psi}_a{}^{\alpha}{}_i\bar{\psi}_b{}^{\beta}{}_j J_{\alpha}{}^{i}{}_{\beta}{}^{j}\big|,
\end{aligned} \tag{25}$$

where $\mathcal{J}_{ab}|$ describe an ordinary space closed two-form.

4. A 2D, $\mathcal{N}$ = 4 density projection operator

It remains for us to calculate the explicit form of the density projection operator (that we will denote by $\Delta^{(4)}$) which is the main purpose of this work. As we are going to describe in this section, by using $\Delta^{(4)}$ and the chiral projector $\overline{\mathcal{D}}^{(4)}$, we can build the integration measure of component actions in 2D, $\mathcal{N} = 4$ minimal supergravity.

As noted by Siegel [17], the 'secret' of the ectoplasmic approach is to realize that the integration theory of superspace can be totally cast into the language of closed super-forms. Indeed it was argued in the work of [2] that the requirement that the topology of a superspace be totally determined by the topology of its purely bosonic sub-manifold naturally provides a reason for the appearance of super-forms in constructing integration measures of superspace.

In the work of [18], it was noted that the derivation of component results follows efficiently from replacing the integration of fermionic coordinates by a process using *first* application of the superspace covariant derivative followed by taking the limit as the Grassmann coordinates are taken to zero. In the 2D, $\mathcal{N} = 4$ case, this is described in the form of an equation

$$\int \mathrm{d}^2\sigma\, \mathrm{d}^4\theta\, \mathrm{d}^4\overline{\theta}\mathrm{E}^{-1}\mathcal{L} \to \int \mathrm{d}^2\sigma \frac{1}{2}\,\mathrm{e}^{-1}[\Delta^{(4)}\overline{\mathcal{D}}^{(4)}\mathcal{L} + \mathrm{h.c.}]| \tag{26}$$

in terms of two differential operators, $\Delta^{(4)}$ (the density projection operator) and $\overline{\mathcal{D}}^{(4)}$ (the chiral projection operator) which may be expanded as

$$\Delta^{(4)} = \sum_{i=0}^{4} b_{(4-i)}\cdot[(\nabla)\times\cdots\times(\nabla)^{4-i}], \tag{27}$$

$$\overline{\mathcal{D}}^{(4)} = \sum_{i=0}^{4} a_{(4-i)}\cdot[(\overline{\nabla})\times\cdots\times(\overline{\nabla})^{4-i}], \tag{28}$$

in terms of some field-dependent coefficients $a_{(4-i)}$ and $b_{(4-i)}$ and powers of the spinorial superspace supergravity covariant derivatives $\nabla_{\alpha i}$ and $\overline{\nabla}_\alpha{}^i$. In (26), we have the expressions $\mathrm{E}^{-1} = \left[\mathrm{Ber}\, E_A{}^M\right]^{-1}$ and $\mathrm{e}^{-1} = \left[\det e_a{}^m\right]^{-1}$ which are functions respectively of the supervielbein and the component vielbein and $\mathrm{d}^2\sigma$ denotes the measure over the two-dimensional bosonic space. A further consequence of (26)–(28) is that the superfield Lagrangian $\mathcal{L}$ need not be hermitian as it is the linear combination of terms that appear in the action formula that must satisfy this requirement. In the present context these spinorial superspace supergravity covariant derivatives satisfy the relations given in section 2.

The basis for the ectoplasmic derivations of local supergravity measures and projections operators lies in a proposition for how to integrate an arbitrary super p-form. This was proposed in the work of [2]. Given a curved superspace with N_B bosonic coordinates (labeled by $\underline{m}$ indices) and N_F fermionic coordinates (labeled by $\underline{\mu}$ indices), we have

Proposition 1. *If $J_{\underline{A}_1\cdots\underline{A}_p}$ is a closed super p-form superfield whose Bianchi identities vanish and $\mathrm{d}\Omega^{\underline{m}_1\cdots\underline{m}_p}$ is a co-chain of dimension $p \leqslant N_B$ (where N_B is the dimensionality of the bosonic subspace), then the integral of the super p-form over the co-chain is given by*

$$\mathcal{S}(\mathrm{d}\Omega|J) \equiv (p!)^{-1} \int \mathrm{d}\Omega^{\underline{a}_1\cdots\underline{a}_p} \mathcal{J}^{(p)}_{\underline{a}_1\cdots\underline{a}_p}\Big|. \tag{29}$$

and this is a supersymmetrical invariant.

In (29) we note that the quantity $\mathcal{J}^{(p)}_{\underline{a}_1\cdots\underline{a}_p}\big|$ is related to the super p-form $J_{\underline{A}_1\cdots\underline{A}_p}$ via

$$\begin{aligned}\left(J_{\underline{a}_1\cdots\underline{a}_p}\big|\right) \equiv \Big[&\mathcal{J}^{(p)}_{\underline{a}_1\cdots\underline{a}_p}\big| + \lambda^{(p,1)}\psi_{[\underline{a}_1|}{}^{\underline{\alpha}_1}\left(J_{\underline{\alpha}_1|\underline{a}_2\cdots\underline{a}_p]}\big|\right) + \lambda^{(p,2)}\psi_{[\underline{a}_1|}{}^{\underline{\alpha}_1}\psi_{|\underline{a}_2|}{}^{\underline{\alpha}_2}\left(J_{\underline{\alpha}_1\underline{\alpha}_2|\underline{a}_3\cdots\underline{a}_p]}\big|\right)\cdots\\ &+ \lambda^{(p,p)}\left[\psi_{\underline{a}_1}{}^{\underline{\alpha}_1}\cdots\psi_{\underline{a}_p}{}^{\underline{\alpha}_p}\right]\left(J_{\underline{\alpha}_1\underline{\alpha}_2\cdots\underline{\alpha}_p}\big|\right)\Big],\end{aligned} \tag{30}$$

where $\psi_{\underline{a}}{}^{\underline{\alpha}}$ denotes the gravitino. The quantities $\mathcal{J}^{(p)}_{\underline{a}_1\cdots\underline{a}_p}\big|$ and coefficients $\lambda^{(p,1)}\cdots\lambda^{(p,p)}$ are determined by taking the limit as the Grassmann coordinates go to zero in $J_{\underline{a}_1\cdots\underline{a}_p}$. In the 2D, $\mathcal{N} = 4$ case with J_{ab} the component of a super two-form, equation (25) informs us about the λ-coefficients.

We next observe that upon setting $p = N_B$ the proposition takes the form

$$\mathcal{S}(\mathrm{d}\Omega|J) = \int \mathrm{d}^{N_B}x\, \mathrm{e}^{-1}\frac{1}{N_B!}\varepsilon^{\underline{a}_1\cdots\underline{a}_{N_B}}\mathcal{J}^{(N_B)}_{\underline{a}_1\cdots\underline{a}_{N_B}}\Big|, \tag{31}$$

where e^{-1} denotes the determinant of the vielbein for the bosonic subspace. In the case considered in this paper, we thus reach the result

$$\begin{aligned}\mathcal{S}(\mathrm{d}\Omega|J) &= \int \mathrm{d}^2\sigma\, \mathrm{e}^{-1}\frac{1}{2}\varepsilon^{ab}\mathcal{J}^{(2)}_{ab}\Big|\\ &= \int \mathrm{d}^2\sigma\, \mathrm{e}^{-1}\Big[\frac{1}{2}\varepsilon^{ab}J_{ab}\big| - \varepsilon^{ab}\left(\psi_a{}^{\alpha i}J_{\alpha i b}\big| + \bar{\psi}_a{}^\alpha{}_i J_\alpha{}^i{}_b\big|\right) - \varepsilon^{ab}\psi_a{}^{\alpha i}\bar{\psi}_b{}^\beta{}_j J_{\alpha i\beta}{}^j\big|\\ &\quad - \frac{1}{2}\varepsilon^{ab}\psi_a{}^{\alpha i}\psi_b{}^{\beta j}J_{\alpha i\beta j}\Big| - \frac{1}{2}\varepsilon^{ab}\bar{\psi}_a{}^\alpha{}_i\bar{\psi}_b{}^\beta{}_j J_\alpha{}^i{}_\beta{}^j\Big|\Big].\end{aligned} \tag{32}$$

More explicitly the equations in (23) are expressed as

$$\begin{aligned}J_{\alpha i\beta}{}^j &= 0,\\ J_{\alpha i\beta j} &= 2(\gamma^3)_{\alpha\beta}\overline{\nabla}^{(2)}_{ij}[\nabla^{(2)\epsilon\kappa} - 2\bar{B}(\gamma^3)^{\epsilon\kappa}]\nabla^{(2)}_{\epsilon\kappa}\overline{\mathcal{L}}\\ &\qquad - C_{\alpha\beta}C_{ij}(\gamma^3)^{\gamma\delta}\overline{\nabla}^{(2)}_{\gamma\delta}[\nabla^{(2)\epsilon\kappa} - 2\bar{B}(\gamma^3)^{\epsilon\kappa}]\nabla^{(2)}_{\epsilon\kappa}\overline{\mathcal{L}},\\ J_\alpha{}^i{}_\beta{}^j &= 2(\gamma^3)_{\alpha\beta}\nabla^{(2)ij}[\overline{\nabla}^{(2)\epsilon\kappa} - 2B(\gamma^3)^{\epsilon\kappa}]\overline{\nabla}^{(2)}_{\epsilon\kappa}\mathcal{L}\end{aligned}$$

$$
\begin{aligned}
&\qquad -C_{\alpha\beta}C^{ij}(\gamma^3)^{\gamma\delta}\nabla^{(2)}_{\gamma\delta}[\overline{\nabla}^{(2)\epsilon\kappa}-2B(\gamma^3)^{\epsilon\kappa}]\overline{\nabla}^{(2)}_{\epsilon\kappa}\mathcal{L},\\
J_{\gamma ka} &= -\frac{\mathrm{i}}{3}\varepsilon_{ab}(\gamma^b)_\gamma{}^\delta\overline{\nabla}_\delta{}^p\overline{\nabla}^{(2)}_{kp}[\nabla^{(2)\epsilon\kappa}-2\bar{B}(\gamma^3)^{\epsilon\kappa}]\nabla^{(2)}_{\epsilon\kappa}\overline{\mathcal{L}},\\
J_\gamma{}^k{}_a &= -\frac{\mathrm{i}}{3}\varepsilon_{ab}(\gamma^b)_\gamma{}^\delta\nabla_{\delta p}\nabla^{(2)kp}[\overline{\nabla}^{(2)\epsilon\kappa}-2B(\gamma^3)^{\epsilon\kappa}]\overline{\nabla}^{(2)}_{\epsilon\kappa}\mathcal{L},\\
J_{ab} &= -\frac{1}{8}\varepsilon_{ab}[\nabla^{(4)}-2\bar{B}(\gamma^3)^{\alpha\beta}\nabla^{(2)}_{\alpha\beta}][\overline{\nabla}^{(2)\epsilon\kappa}-2B(\gamma^3)^{\epsilon\kappa}]\overline{\nabla}^{(2)}_{\epsilon\kappa}\mathcal{L}\\
&\qquad -\frac{1}{8}\varepsilon_{ab}[\overline{\nabla}^{(4)}-2B(\gamma^3)^{\alpha\beta}\overline{\nabla}^{(2)}_{\alpha\beta}][\nabla^{(2)\epsilon\kappa}-2\bar{B}(\gamma^3)^{\epsilon\kappa}]\nabla^{(2)}_{\epsilon\kappa}\overline{\mathcal{L}}.
\end{aligned}
\tag{33}
$$

Finally, the results in (33) can be substituted into equation (32) to reach the main result of this presentation. Given an arbitrary 2D, $\mathcal{N}=4$ superfield Lagrangian $\mathcal{L}$, a local supersymmetrical invariant is given by

$$
\begin{aligned}
\mathcal{S} &= \int \mathrm{d}^2\sigma\,\mathrm{e}^{-1}\Delta^{(4)}\bar{\mathcal{D}}^{(4)}\mathcal{L}\Big| + \text{h.c.}\\
&= \int \mathrm{d}^2\sigma\mathrm{e}^{-1}\Big\{\frac{1}{8}\nabla^{(4)}-\frac{1}{4}\bar{B}(\gamma^3)^{\alpha\beta}\nabla^{(2)}_{\alpha\beta}+\frac{\mathrm{i}}{3}\bar{\psi}_a{}^\gamma{}_i(\gamma^a)_\gamma{}^\delta\nabla_{\delta j}\nabla^{(2)ij}\\
&\quad -\varepsilon^{ab}\bar{\psi}_a{}^\alpha{}_i\bar{\psi}_b{}^\beta{}_j(\gamma^3)_{\alpha\beta}\nabla^{(2)ij}+\frac{1}{2}\varepsilon^{ab}\bar{\psi}_a{}^\alpha{}_i\bar{\psi}_b{}^\beta{}_jC_{\alpha\beta}C^{ij}(\gamma^3)^{\gamma\delta}\nabla^{(2)}_{\gamma\delta}\Big\}\\
&\quad\times[\overline{\nabla}^{(2)\epsilon\kappa}-2B(\gamma^3)^{\epsilon\kappa}]\overline{\nabla}^{(2)}_{\epsilon\kappa}\mathcal{L}\Big| + \text{h.c.}
\end{aligned}
\tag{34}
$$

in the presence of the off-shell supergravity theory described in section 2.

5. Conclusion

With this present work, we have completed the task of developing an efficient local superspace integration theory for two-dimensional theories that possess eight real supercharges. We believe that the result given in (34) is unexpectedly elegant and simple given that the general form of the eighth-order spinorial differential operator defined by (26), (27) and (28) could, in principle, take a more complicated form. Perhaps one of most surprising features of this derivation has been the use of the closed 2D, $\mathcal{N}=4$ super two-form used in theorem 1 The superfield U that appeared in equation (23) is *not* required to describe any irreducible supermultiplet. The only requirement imposed on the superfield U is its chirality.

As proved in [21], the chiral superfield U can be expressed in terms of the chiral projector $\overline{\mathcal{D}}^{(4)}$ and an unconstrained superfield $\mathcal{L}$ as $U=\overline{\mathcal{D}}^{(4)}\mathcal{L}$. This result has been used in sections 3 and 4. According to the discussion of section 4, the main result of this paper is the computation of the density projector operator $\Delta^{(4)}$ of (26), (27) and (34), which, together with $\overline{\mathcal{D}}^{(4)}$, allows to define the component supergravity integration measure (34). In deriving for the first time $\Delta^{(4)}$ we used the ectoplasmic techniques and the new super two-form of theorem 1 (23).

One other point we wish to emphasize is the efficiency of the ectoplasmic approach in the case we considered here. It would be interesting to re-derive the integration measure (34) via the normal coordinate expansion technique [7, 15, 16] (in particular using its last version [15]) even if we do not expect that the latter approach would require shorter computations. This is especially true considering that in 2D the number of Bianchi identities to be solved for a closed super two-form is relatively low. This emphasizes again the important role of forms as a basis for superspace integration theory as advocated in the ectoplasmic approach. The success of this also points to the generality of using this as a tool in all cases to derive superspace local integration measures.

Note also that here we focused on the 2D, $\mathcal{N} = 4$ minimal superspace geometry of [1] as described in section 2. In general, it is known that there could exist different off-shell superspace supergravities. We expect that the ectoplasm paradigm and the results of our paper can be extended to any covariant superspace formulation of 2D, $\mathcal{N} = 4$ supergravity. For example, in the first paper of [1] a variant central charge formulation of the minimal multiplet was given; once noted that the Lagrangian $\mathcal{L}$ in (34) has to be neutral for the central charges, one can see that the results of our paper apply without modifications to the variant formulation. Moreover, recently a new extended covariant formulation of 2D, $\mathcal{N} = 4$ supergravity in superspace was given [24]. The ectoplasm techniques to compute the chiral action in components apply straightforwardly if one consider the geometry of [24] even if in this case longer computations are expected due to the more involved structure of the torsion multiplet. Other superspace formulations of 2D, $\mathcal{N} = 4$ supergravity [25] are known in the bi-harmonic superspace of [26]. Being those superspace supergravities based on a prepotential approach, the definition of a covariant components reduction is not clear. However, on the ground of the related bi-projective formalism [27], recently extended to covariantly study 2D, $\mathcal{N} = 4$ matter-couplet supergravity, it would be of interest and well defined to find by using ectoplasm techniques, the bi-projective density operator analogously to the chiral action studied here.

'Where the senses fail us, reason must step in.'
Galileo Galilei

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Appendix A. consistency of Bianchi identities and constraints for two-form

In this appendix, we will present the explicit proof that the Bianchi identities associated with the results in (23) imply that it is a closed 2D, $\mathcal{N} = 4$ super two-form. We begin by writing an ansatz for the lowest components of a 2D, $\mathcal{N} = 4$ super two-form under the assumption that these component should

(a) be linear in a (anti)chiral superfield U $(\bar{U})$; $\overline{\nabla}_\alpha{}^i U = 0$ $(\nabla_{\alpha i}\bar{U} = 0)$,
(b) depend on the superspace supergravity covariant derivative,
(c) be local functions of the superspace supergravity field strengths B, $\overline{B}$, G and H.

Under the previous assumptions we will begin with an ansatz given by[6]

$$J_{\alpha i\,\beta j} = a(\gamma^3)_{\alpha\beta}\overline{\nabla}^{(2)}_{ij}\bar{U} + bC_{\alpha\beta}C_{ij}(\gamma^3)^{\gamma\delta}\overline{\nabla}^{(2)}_{\gamma\delta}\bar{U} + C_{\alpha\beta}C_{ij}F\bar{U}, \tag{A.1}$$

$$J_{\alpha i\,\beta}{}^{j} = 0, \qquad J_\alpha{}^i{}_\beta{}^j = -(J_{\alpha i\,\beta j})^*, \tag{A.2}$$

[6] The ansatz we are using can also be guessed by (i) considering the flat 4D, $\mathcal{N} = 2$ 'chiral' closed super four-form introduced in [23]; (ii) performing a dimensional reduction of the 4D, $\mathcal{N} = 2$ super four-form to derive a 2D, $\mathcal{N} = 4$ closed super two-form; (iii) extending the resulting dimension-1 components of the flat 2D, $\mathcal{N} = 4$ two-form to the curved case by modifying the flat derivatives to the curved covariant derivatives and by adding torsion-dependent terms.

where

$$F = F(B, \bar{B}, G, H) = b_1 B + b_2 \bar{B} + gG + hH, \tag{A.3}$$

and $a,\ b,\ b_1,\ b_2,\ g,\ h$ are constants to be fixed.

The task is to study the Bianchi identities that derive from the closure of the two-form J:

$$\mathrm{d}J = 0, \qquad \Longleftrightarrow 0 = \tfrac{1}{2}\nabla_{[A}J_{BC)} - \tfrac{1}{2}T_{[AB|}{}^{D}J_{D|C)}, \tag{A.4}$$

with $J_{AB} = \big(J_{\alpha i\beta j},\ J_{\alpha i\beta}{}^{j},\ J_{\alpha}{}^{i}{}_{\beta}{}^{j},\ J_{\gamma ka},\ J_{\gamma}{}^{k}{}_{a},\ J_{ab}\big)$ and the lowest components satisfying (A.1)–(A.3).

Substituting the results of (A.1), (A.2) into the identity (A.4) with $A = \alpha i,\ B = \beta j,\ C = \gamma k$ one obtains

$$\begin{aligned}0 = {}& a(\gamma^3)_{\beta\gamma}\big[\nabla_{\alpha i}, \overline{\nabla}^{(2)}_{jk}\big]\overline{U} + a(\gamma^3)_{\gamma\alpha}\big[\nabla_{\beta j}, \overline{\nabla}^{(2)}_{ki}\big]\overline{U} + a(\gamma^3)_{\alpha\beta}\big[\nabla_{\gamma k}, \overline{\nabla}^{(2)}_{ij}\big]\overline{U}\\ &+ bC_{\beta\gamma}C_{jk}(\gamma^3)^{\delta\rho}\big[\nabla_{\alpha i}, \overline{\nabla}^{(2)}_{\delta\rho}\big]\overline{U} + bC_{\gamma\alpha}C_{ki}(\gamma^3)^{\delta\rho}\big[\nabla_{\beta j}, \overline{\nabla}^{(2)}_{\delta\rho}\big]\overline{U}\\ &+ bC_{\alpha\beta}C_{ij}(\gamma^3)^{\delta\rho}\big[\nabla_{\gamma k}, \overline{\nabla}^{(2)}_{\delta\rho}\big]\overline{U} + C_{\beta\gamma}C_{jk}(\nabla_{\alpha i}F)\overline{U} + C_{\gamma\alpha}C_{ki}(\nabla_{\beta j}F)\overline{U}\\ &+ C_{\alpha\beta}C_{ij}(\nabla_{\gamma k}F)\overline{U},\end{aligned} \tag{A.5}$$

where we have used the fact that $\bar{U}$ is antichiral to write this. At this point, there are two useful identities to note

$$\big[\nabla_{\alpha i}, \overline{\nabla}^{(2)}_{ij}\big]\overline{U} = (-2\mathrm{i}C_{i(j}(\gamma^c)_{\alpha}{}^{\delta}\nabla_c\overline{\nabla}_{\delta k)})\overline{U} \tag{A.6}$$

$$\big[\nabla_{\alpha i}, (\gamma^3)^{\delta\rho}\overline{\nabla}^{(2)}_{\delta\rho}\big] = (-4\mathrm{i}\varepsilon^{bc}(\gamma_b)_{\alpha}{}^{\beta}\nabla_c\overline{\nabla}_{\beta i})\overline{U} \tag{A.7}$$

which shows that in principle there are terms containing spacetime derivatives in (A.5). In order to satisfy the Bianchi identity, two sets of conditions are required:

$$\begin{aligned}&\text{(a)}\quad a = -2b \quad \text{and}\\ &\text{(b)}\quad b_1 = b_2 = g = h = 0.\end{aligned} \tag{A.8}$$

For simplicity we also set

$$a = 1. \tag{A.9}$$

The next Bianchi identity encountered takes the form

$$0 = \overline{\nabla}^{i}_{\alpha}J_{\beta j\gamma k} + T_{\alpha}{}^{i}{}_{\beta j}{}^{a}J_{\gamma ka} + T_{\alpha}{}^{i}{}_{\gamma k}{}^{a}J_{\beta ja}. \tag{A.10}$$

The result in (A.1), subject to (A.8), (A.9), can be substituted into this equation. To satisfy this, it is useful to use the following identities:

$$\begin{aligned}&\overline{\nabla}_{\alpha i}\overline{\nabla}^{(2)}_{jk}\bar{U} = -\tfrac{1}{3}C_{i(j}\overline{\nabla}^{p}_{\alpha}\overline{\nabla}^{(2)}_{k)p}\overline{U},\\ &\overline{\nabla}_{\alpha i}\overline{\nabla}^{(2)}_{\beta\gamma}\bar{U} = \tfrac{1}{3}C_{\alpha(\beta}\overline{\nabla}^{p}_{\gamma)}\overline{\nabla}^{(2)}_{ip}\overline{U} - \tfrac{4}{3}BC_{\alpha(\beta}(\gamma^3)_{\gamma)}{}^{\delta}\overline{\nabla}_{\delta i}\overline{U} + \tfrac{1}{3}B(\gamma^3)_{(\alpha\beta}\overline{\nabla}_{\gamma)i}\overline{U},\\ &(\gamma^3)^{\beta\gamma}\overline{\nabla}_{\alpha i}\overline{\nabla}^{(2)}_{\beta\gamma}\bar{U} = -\tfrac{2}{3}(\gamma^3)_{\alpha}{}^{\gamma}\overline{\nabla}^{p}_{\gamma}\overline{\nabla}^{(2)}_{ip}\overline{U}.\end{aligned} \tag{A.11}$$

Then, to completely satisfy (A.10) one has to impose

$$J_{\gamma ka} = -\frac{\mathrm{i}}{3}\varepsilon_{ab}(\gamma^b)_{\gamma}{}^{\rho}\overline{\nabla}_{\rho}{}^{p}\overline{\nabla}^{(2)}_{kp}\bar{U}. \tag{A.12}$$

Note that it holds

$$J_{\gamma}{}^{k}{}_{a} = -(J_{\gamma ka})^{*} = -\frac{\mathrm{i}}{3}\varepsilon_{ab}(\gamma^b)_{\gamma}{}^{\rho}\nabla_{\rho p}\nabla^{(2)kp}U. \tag{A.13}$$

We can continue our deliberations by considering the Bianchi identity given by

$$0 = \nabla_a J_{\beta j\gamma k} + \nabla_{\beta j} J_{\gamma k a} + \nabla_{\gamma k} J_{\beta j a} - T_{a\beta j}{}^{\delta l} J_{\delta l \gamma k} - T_{a\gamma k}{}^{\delta l} J_{\delta l \beta j}, \tag{A.14}$$

and into this are substituted the results (A.1), (A.8), (A.9) and (A.12). When this is done, a differential equation on $\overline{U}$ of the form

$$\begin{aligned} 0 = {} & \nabla_a\big(2(\gamma^3)_{\beta\gamma}\overline{\nabla}^{(2)}_{jk} - C_{\beta\gamma}C_{jk}(\gamma^3)^{\delta\rho}\overline{\nabla}^{(2)}_{\delta\rho}\big)\overline{U} \\ & - \frac{\mathrm{i}}{3}\nabla_{\beta j}\big(\varepsilon_{ab}(\gamma^b)_\gamma{}^\delta\overline{\nabla}_\delta{}^p\overline{\nabla}^{(2)}_{kp}\overline{U}\big) - \frac{\mathrm{i}}{3}\nabla_{\gamma k}\big(\varepsilon_{ab}(\gamma^b)_\beta{}^\delta\overline{\nabla}_\delta{}^p\overline{\nabla}^{(2)}_{jp}\overline{U}\big) \\ & + \frac{\mathrm{i}}{2}\delta^l_j\phi_\beta{}^\rho(\gamma_a)_\rho{}^\delta\big(2(\gamma^3)_{\delta\gamma}\overline{\nabla}^{(2)}_{lk} - C_{\delta\gamma}C_{lk}(\gamma^3)^{\rho\tau}\overline{\nabla}^{(2)}_{\rho\tau}\big)\overline{U} \\ & + \frac{\mathrm{i}}{2}\delta^l_k\phi_\gamma{}^\rho(\gamma_a)_\rho{}^\delta\big(2(\gamma^3)_{\delta\beta}\overline{\nabla}^{(2)}_{lj} - C_{\delta\beta}C_{lj}(\gamma^3)^{\rho\tau}\overline{\nabla}^{(2)}_{\rho\tau}\big)\overline{U} \end{aligned} \tag{A.15}$$

emerges. Further progress is possible by using the identity

$$\begin{aligned} \big\{\nabla_{\alpha i}, \overline{\nabla}^p_\delta\overline{\nabla}^{(2)}_{kp}\big\}\overline{U} = {} & \big(3\mathrm{i}(\gamma^a)_{\alpha\delta}\nabla_a\overline{\nabla}^{(2)}_{ik} - 3\mathrm{i}C_{ik}(\gamma^a)_\alpha{}^\rho\nabla_a\overline{\nabla}^{(2)}_{\delta\rho} \\ & + \tfrac{3}{2}C_{ik}\phi^\tau{}_\delta(\gamma^3)_{\alpha\tau}(\gamma^3)^{\rho\beta}\overline{\nabla}^{(2)}_{\beta\rho} - 3\phi_{\alpha\delta}\overline{\nabla}^{(2)}_{ik} \\ & + 6C_{ik}C_{\alpha\delta}\Sigma^{\beta p}\overline{\nabla}_{\beta p} + 6C_{ik}(\gamma^3)_{\alpha\delta}(\gamma^3)^{\beta\rho}\Sigma_\beta{}^p\overline{\nabla}_{\rho p}\big)\overline{U}. \end{aligned} \tag{A.16}$$

This result is substituted into (A.15) and after some algebra, the Σ-dependent terms are seen to cancel leaving

$$\begin{aligned} 0 = {} & \big(2(\gamma^3)_{\alpha\gamma}\nabla_a\overline{\nabla}^{(2)}_{ik} - C_{\alpha\gamma}C_{ik}(\gamma^3)^{\delta\rho}\nabla_a\overline{\nabla}^{(2)}_{\delta\rho} - 2(\gamma^3)_{\alpha\gamma}\nabla_a\overline{\nabla}^{(2)}_{ik} + C_{ik}C_{\alpha\gamma}(\gamma^3)^{\delta\rho}\nabla_a\overline{\nabla}^{(2)}_{\delta\rho} \\ & - \mathrm{i}\varepsilon_{ab}\phi_{\beta\delta}(\gamma^b)_{\alpha\gamma}C^{\beta\delta}\overline{\nabla}^{(2)}_{ik} - \mathrm{i}\phi_{\beta\delta}(\gamma^3)^{\beta\delta}(\gamma_a)_{\alpha\gamma}\overline{\nabla}^{(2)}_{ik} \\ & + \mathrm{i}\varepsilon_{ab}\phi_{\alpha'\delta}(\gamma^b)_{\alpha\delta}C^{\alpha'\delta}\overline{\nabla}^{(2)}_{ik} + \mathrm{i}\phi_{\alpha'\delta}(\gamma_c)_{\alpha\delta}(\gamma^3)^{\alpha'\delta}\overline{\nabla}^{(2)}_{ik}\big)\overline{U} \end{aligned} \tag{A.17}$$

which is clearly identically satisfied.

There is a second dimension-2 Bianchi identity of the form

$$0 = -\overline{\nabla}_\alpha{}^i J_{\gamma k b} - \nabla_{\gamma k} J_\alpha{}^i{}_b + T_{b\gamma k}{}^\delta{}_l J_\delta{}^l{}_\alpha{}^i + T_{b\alpha}{}^{i\delta l} J_{\delta l \gamma k} + T_{\gamma k\alpha}{}^{ic} J_{cb}. \tag{A.18}$$

One may substitute from results derived previously to cast this into the form of

$$\begin{aligned} 2\mathrm{i}\delta^i_k(\gamma^c)_{\alpha\gamma}J_{bc} = {} & \mathrm{i}\big(\tfrac{1}{3}\varepsilon_{bc}(\gamma^c)_\gamma{}^\rho\overline{\nabla}_\alpha{}^i\overline{\nabla}_\rho{}^p\overline{\nabla}^{(2)}_{kp} + B(\gamma_b)_{\alpha\gamma}\overline{\nabla}^{(2)i}{}_k - \tfrac{1}{2}B\delta^i_k(\gamma^3\gamma_b)_{\alpha\gamma}(\gamma^3)^{\rho\tau}\overline{\nabla}^{(2)}_{\rho\tau}\big)\overline{U} \\ & - \nabla_{\gamma k}J_\alpha{}^i{}_b + T_{b\gamma k}{}^\delta{}_l J_\delta{}^l{}_\alpha{}^i, \end{aligned} \tag{A.19}$$

and progress is achieved in analyzing this identity by noting that it holds

$$\overline{\nabla}_\alpha{}^i\overline{\nabla}_\beta{}^k\overline{\nabla}^{(2)}_{jk}\overline{U} = \big(\tfrac{1}{2}C_{\alpha\beta}\overline{\nabla}^{(2)ik}\overline{\nabla}^{(2)}_{jk} + \tfrac{1}{2}C^{ik}\overline{\nabla}^{(2)}_{\alpha\beta}\overline{\nabla}^{(2)}_{jk} - 2B(\gamma^3)_{\alpha\beta}C^{ip}\overline{\nabla}^{(2)}_{pj}\big)\bar{U}. \tag{A.20}$$

One other identity tells us

$$\overline{\nabla}^{(2)}_{\alpha\beta}\overline{\nabla}^{(2)}_{ij}\bar{U} = -2B(\gamma^3)_{\alpha\beta}\overline{\nabla}^{(2)}_{ij}\overline{U}, \tag{A.21}$$

so that (A.20) becomes

$$\begin{aligned} \overline{\nabla}_\alpha{}^i\overline{\nabla}_\beta{}^k\overline{\nabla}^{(2)}_{jk}\overline{U} & = \big(\tfrac{1}{2}C_{\alpha\beta}\overline{\nabla}^{(2)ik}\overline{\nabla}^{(2)}_{jk} - 3B(\gamma^3)_{\alpha\beta}C^{ip}\overline{\nabla}^{(2)}_{pj}\big)\overline{U} \\ & = \big(\tfrac{1}{4}\delta_j{}^iC_{\alpha\beta}\overline{\nabla}^{(2)kl}\overline{\nabla}^{(2)}_{kl} - 3B(\gamma^3)_{\alpha\beta}C^{ip}\overline{\nabla}^{(2)}_{pj}\big)\overline{U} \\ & = \big(\tfrac{3}{4}\delta_j{}^iC_{\alpha\beta}\overline{\nabla}^{(4)} - 3B(\gamma^3)_{\alpha\beta}C^{ip}\overline{\nabla}^{(2)}_{pj}\big)\overline{U}, \end{aligned} \tag{A.22}$$

where on the first term we have used a sequence of identities (see also the final appendix). The final line of (A.22) can now be substituted into (A.19) to yield after a bit of algebra

$$\begin{aligned} 2\mathrm{i}\delta^i_k(\gamma^c)_{\alpha\gamma}J_{bc} = {} & \mathrm{i}\big(-\tfrac{1}{4}\varepsilon_{bc}(\gamma^c)_{\alpha\gamma}\delta^i_k\overline{\nabla}^{(4)} + \tfrac{1}{2}B\varepsilon_{bc}\delta^i_k(\gamma^c)_{\alpha\gamma}(\gamma^3)^{\rho\tau}\overline{\nabla}^{(2)}_{\rho\tau}\big)\overline{U} \\ & - \nabla_{\gamma k}J_\alpha{}^i{}_b + T_{b\gamma k}{}^\delta{}_l J_\delta{}^l{}_\alpha{}^i. \end{aligned} \tag{A.23}$$

Finally this equation informs us that

$$J_{ab} = \varepsilon_{ab}\big(-\tfrac{1}{8}\overline{\nabla}^{(4)} + \tfrac{1}{4}B(\gamma^3)^{\alpha\beta}\overline{\nabla}^{(2)}_{\alpha\beta}\big)\overline{U} + \text{h.c.} \tag{A.24}$$

There remains one final Bianchi identity of the form

$$\begin{aligned} 0 &= \nabla_{\alpha i}J_{bc} - \nabla_b J_{\alpha i c} + \nabla_c J_{\alpha i b} + T_{\alpha i b}{}^D J_{Dc} + T_{\alpha i c}{}^D J_{Db} - T_{bc}{}^{\delta l} J_{\delta l \alpha i}, \\ 0 &= \varepsilon^{ab}\big(\nabla_{\alpha i}J_{ab} + 2\nabla_a J_{b\alpha i} - 2T_{\alpha i a}{}^{\delta l}J_{\delta l b} - 2T_{\alpha i a}{}^{\delta}{}_l J_{\delta}{}^l{}_b - T_{ab}{}^{\delta l}J_{\delta l \alpha i}\big). \end{aligned} \tag{A.25}$$

To prove that this identity is satisfied requires a calculation of some length. The key to its satisfaction requires one final identity

$$\begin{aligned} [\nabla_{\alpha i}, \overline{\nabla}^{(4)}]\bar{U} = \bigg(&-\frac{8i}{3}(\gamma^a)_\alpha{}^\rho\nabla_a\overline{\nabla}_\rho{}^p\overline{\nabla}^{(2)}_{ip} - 8\mathrm{i}B\varepsilon_{bc}(\gamma^b)_\alpha{}^\beta\nabla^c\overline{\nabla}_{\beta i} \\ &+ \frac{8}{3}\phi_\alpha{}^\gamma\overline{\nabla}_\gamma{}^p\overline{\nabla}^{(2)}_{ip} + 8\Sigma_\alpha{}^l\overline{\nabla}^{(2)}_{il}\bigg)\overline{U} \end{aligned} \tag{A.26}$$

that is valid for the supergravity covariant derivative acting on an antichiral scalar superfield such as $\bar{U}$.

Other Bianchi identities, not explicitly mentioned here, are identically solved by complex conjugation of the results obtained in this section.

Appendix B. Miscellaneous identities

For the reader's convenience, here we also collect some useful formulas used in the derivations provided in this paper and especially in appendix A (we recall that $\bar{U}$ is antichiral)

$$\overline{\nabla}_\alpha{}^i\overline{\nabla}_\beta{}^j = \frac{1}{2}C_{\alpha\beta}\overline{\nabla}^{(2)ij} + \frac{1}{2}C^{ij}\overline{\nabla}^{(2)}_{\alpha\beta} + BC_{\alpha\beta}C^{ij}\mathcal{M} - B(\gamma^3)_{\alpha\beta}\mathcal{Y}^{ij}, \tag{B.1}$$

$$\big[\nabla_{\alpha i}, \overline{\nabla}^{(2)}_{jk}\big]\overline{U} = -2\mathrm{i}C_{i(j}(\gamma^c)_\alpha{}^\delta\nabla_c\overline{\nabla}_{\delta k)}\overline{U}, \tag{B.2}$$

$$\begin{aligned} \big[\nabla_{\alpha i}, \overline{\nabla}^{(2)}_{\delta\rho}\big]\overline{U} = \bigg(&-2\mathrm{i}(\gamma^c)_{\alpha(\delta}\nabla_c\overline{\nabla}_{\rho)i} - G(\gamma^3)_{\alpha(\delta}(\gamma^3)_{\rho)}{}^\gamma\overline{\nabla}_{\gamma i} + GC_{\alpha(\delta}\overline{\nabla}_{\rho)i} \\ &- \frac{\mathrm{i}}{2}HC_{\alpha(\delta}(\gamma^3)_{\rho)}{}^\tau\overline{\nabla}_{\tau i} + \frac{\mathrm{i}}{2}H(\gamma^3)_{\alpha(\delta}\overline{\nabla}_{\rho)i} - \mathrm{i}H(\gamma^3)_{\delta\rho}\overline{\nabla}_{\alpha i}\bigg)\overline{U}, \end{aligned} \tag{B.3}$$

$$\big[\nabla_{\alpha i}, (\gamma^3)^{\delta\rho}\overline{\nabla}^{(2)}_{\delta\rho}\big]\overline{U} = -4\mathrm{i}\varepsilon^{bc}(\gamma_b)_\alpha{}^\beta\nabla_c\overline{\nabla}_{\beta i}\overline{U}, \tag{B.4}$$

$$\overline{\nabla}_{\alpha i}\overline{\nabla}^{(2)}_{jk}\overline{U} = -\frac{1}{3}C_{i(j}\overline{\nabla}_\alpha{}^p\overline{\nabla}^{(2)}_{k)p}\overline{U}, \tag{B.5}$$

$$\overline{\nabla}_{\alpha i}\overline{\nabla}^{(2)}_{\beta\gamma}\overline{U} = \frac{1}{3}C_{\alpha(\beta}\overline{\nabla}_{\gamma)}{}^p\overline{\nabla}^{(2)}_{ip}\overline{U} - \frac{4}{3}BC_{\alpha(\beta}(\gamma^3)_{\gamma)}{}^\delta\overline{\nabla}_{\delta i}\overline{U} + \frac{1}{3}B(\gamma^3)_{(\alpha\beta}\overline{\nabla}_{\gamma)i}\overline{U}, \tag{B.6}$$

$$(\gamma^3)^{\beta\gamma}\overline{\nabla}_{\alpha i}\overline{\nabla}^{(2)}_{\beta\gamma}\overline{U} = -\frac{2}{3}(\gamma^3)_\alpha{}^\gamma\overline{\nabla}_\gamma{}^p\overline{\nabla}^{(2)}_{ip}\overline{U}, \tag{B.7}$$

$$\overline{\nabla}^\gamma{}_i\overline{\nabla}^{(2)}_{\alpha\gamma}\overline{U} = -\overline{\nabla}_\alpha{}^p\overline{\nabla}^{(2)}_{ip}\overline{U} + 4B(\gamma^3)_\alpha{}^\delta\overline{\nabla}_{\delta i}\overline{U}, \tag{B.8}$$

$$\begin{aligned} \big\{\nabla_{\alpha i}, \overline{\nabla}_\delta{}^p\overline{\nabla}^{(2)}_{kp}\big\}\overline{U} = \bigg(&3\mathrm{i}(\gamma^a)_{\alpha\delta}\nabla_a\overline{\nabla}^{(2)}_{ik} - 3\mathrm{i}C_{ik}(\gamma^a)_\alpha{}^\rho\nabla_a\overline{\nabla}^{(2)}_{\delta\rho} \\ &+ \frac{3}{2}C_{ik}\phi^\tau{}_\delta(\gamma^3)_{\alpha\tau}(\gamma^3)^{\rho\beta}\overline{\nabla}^{(2)}_{\beta\rho} - 3\phi_{\alpha\delta}\overline{\nabla}^{(2)}_{ik} \\ &+ 6C_{ik}C_{\alpha\delta}\Sigma^{\beta p}\overline{\nabla}_{\beta p} + 6C_{ik}(\gamma^3)_{\alpha\delta}(\gamma^3)^{\beta\gamma}\Sigma_\beta{}^p\overline{\nabla}_{\gamma p}\bigg)\overline{U}, \end{aligned} \tag{B.9}$$

$$\overline{\nabla}^{(2)}_{\alpha\beta}\overline{\nabla}^{(2)}_{ij}\overline{U} = -2B(\gamma^3)_{\alpha\beta}\overline{\nabla}^{(2)}_{ij}\overline{U}, \tag{B.10}$$

$$\overline{\nabla}^{(2)\alpha\beta}\overline{\nabla}^{(2)}_{\alpha\beta}\overline{U} = -\overline{\nabla}^{(2)ij}\overline{\nabla}^{(2)}_{ij}\overline{U} - 4B(\gamma^3)^{\alpha\beta}\overline{\nabla}^{(2)}_{\alpha\beta}\overline{U}, \tag{B.11}$$

$$\overline{\nabla}^{(4)}\overline{U} := -\frac{1}{3}\overline{\nabla}^{(2)kl}\overline{\nabla}^{(2)}_{kl}\overline{U}, \tag{B.12}$$

$$\overline{\nabla}_\alpha{}^i\overline{\nabla}_\beta{}^k\overline{\nabla}^{(2)}_{jk}\overline{U} = \left(\frac{3}{4}C_{\alpha\beta}\delta^i_j\overline{\nabla}^{(4)} - 3B(\gamma^3)_{\alpha\beta}\overline{\nabla}^{(2)i}{}_j\right)\overline{U}, \tag{B.13}$$

$$\overline{\nabla}_\alpha{}^i(\overline{\nabla}^{(2)\gamma\delta} - 2B(\gamma^3)^{\gamma\delta})\overline{\nabla}^{(2)}_{\gamma\delta}\overline{U} = 0, \tag{B.14}$$

$$\begin{aligned}[\nabla_{\alpha i}, \overline{\nabla}^{(4)}]\overline{U} = \bigg(&-\frac{8i}{3}(\gamma^c)_\alpha{}^\beta\nabla_c\overline{\nabla}_\beta{}^k\overline{\nabla}^{(2)}_{ik} - 8iB\varepsilon^{ab}(\gamma_a)_\alpha{}^\delta\nabla_b\overline{\nabla}_{\delta i} \\ &+8\Sigma_\alpha{}^j\overline{\nabla}^{(2)}_{ij} + \frac{8}{3}\phi_\alpha{}^\gamma\overline{\nabla}_\gamma{}^k\overline{\nabla}^{(2)}_{ik}\bigg)\overline{U}.\end{aligned} \tag{B.15}$$

By complex conjugation, the reader can derive an analog set of equations for the chiral superfield U.

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